Tropical totally positive Grassmannian

(Speyer–Williams 2003)

$$R = \bigcup_{n=1}^{\infty} \mathbb{R}((t^{1/n}))$$

$$\mathcal{C} = \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$$ field of Puiseaux series

Ex $f = \sum_{k=k_0}^{\infty} c_k t^{k/n}$ $\quad$ $val(f) = k_0/n$

$$R^+ = \left\{ g(t) \in \mathcal{C} \;\middle|\; \begin{array}{l}\text{coefficient of lowest} \\ \text{term is real \& positive}\end{array} \right\}$$

Ex $g(t) = \underbrace{3}_{\in \mathbb{R}^+} t^{-7/2} + t^{-5/2} - (3+i)\, t^{-3/2}$

Thm/Def

Let $I \subset \mathcal{C}[x_1, \dots, x_n]$ be an ideal.

Then, $Trop(V(I))$ is

(i) $\overline{\underbrace{val}_{\text{coordinate-wise valuation}}\left(V(I) \cap (\mathcal{C}^*)^n\right)} \subset \mathbb{R}^n$ $\qquad$ $in_\omega(f) = \sum_{\substack{u : val(c_u) + u\cdot w \\ = trop(f)(w)}} \overline{t^{-val(c_u)} c_u}\, x^u \in \mathbb{K}[x^{\pm}]$

(ii) $\{ \omega \in \mathbb{R}^n \mid in_\omega(I)$ contains no monomials $\}$

Def $Trop^+ V(I) = \overline{val\left(V(I) \cap (R^+)^n\right)}$

Thm (SW '03)

$$Trop^+ V(I) = \left\{ \omega \in \mathbb{R}^n \;\middle|\; \begin{array}{l} in_\omega(I) \text{ contains no} \\ \text{non-zero polynomials in } \mathbb{R}^+[x_1, \dots, x_n] \end{array} \right\}$$

Cor $\mathrm{Trop}^+ V(I)$ is a subcomplex of the polyhedral complex $\mathrm{Trop}\, V(I)$

Ex $f(x,y) = x-y+1 \in \mathbb{C}[x,y]$

$V(f) \cap (\mathbb{C}^*)^2 = \{(z, z+1) \mid z \in \mathbb{C} \setminus \{0,-1\}\}$ $\quad V(f) \cap (\mathbb{R}^+)^2 = \{(z, z+1) \mid z \in \mathbb{R}^+\}$

$$(\mathrm{val}(z), \mathrm{val}(z+1)) = \begin{cases} (\mathrm{val}\, z, 0) & \mathrm{val}\, z > 0 \\ (\mathrm{val}\, z, \mathrm{val}\, z) & \mathrm{val}\, z < 0 \\ (0, \mathrm{val}(z+1)) & \mathrm{val}\, z = 0,\ \mathrm{val}\, z+1 > 0 \\ (0, 0) & \text{else} \end{cases}$$

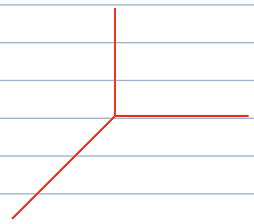
$\to z = t^a \quad a \in \mathbb{R}^+ \quad (\mathrm{val}(t^a), \mathrm{val}(t^a+1)) = (a, 0)$

$\to z = t^b \quad b \in \mathbb{R}^-$

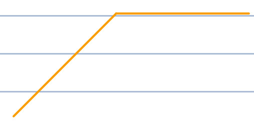
$\to z = ct^0 + t^{\varepsilon}(\sim\sim\sim)$

$z+1 = \underbrace{(c+1)}_{=0} t^0 + t^{\varepsilon}(\sim\sim)$

$c = -1$ a contradiction



$\mathrm{Trop}(x-y+1)$



$\mathrm{Trop}^+(x-y+1) = \mathrm{val}(\{(z, z+1) \mid z \in \mathbb{R}^+\})$

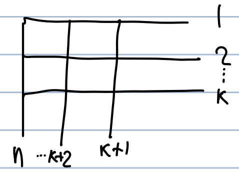
Note: val looks at exponents not values

$$\mathrm{in}_{1,0}(x-y+1) = \mathrm{in}_{10}(x^1y^0 - x^0y^1 + x^0y^0)$$
$$= -y+1$$

$\mathrm{in}_{-1,-1}(x-y+1) = x+1$ issue because all coeffs positive

$\mathrm{in}_{0,0}(x-y+1) = x-y+1$

§ Parametrizing the Grassmannian

Web 

Let $I \in \binom{[n]}{k}$

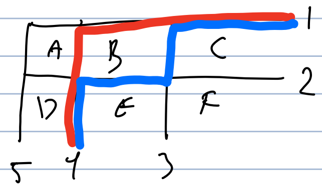
Path $I = \left\{ \begin{array}{l} \text{pairwise disjoint paths from} \\ [k] \setminus I \text{ to } I \setminus [k] \end{array} \right\}$

$$\text{prod } S = \prod_{\substack{\text{paths} \\ p \in S}} \prod_{\substack{\text{regions } r \\ \text{below } p}} x_r$$

$S \in \text{path } I$

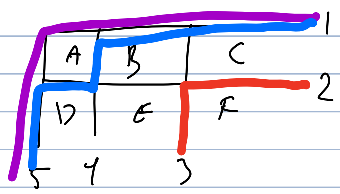
Thm Have bijection $(\mathbb{R}^+)^{k(n-k)} \to Gr_{k,n}(\mathbb{R}^+)$

by $\Delta_I = \sum_{S \in \text{path } I} \text{Prod } S$

Ex 

path(24) = paths $1 \leftrightarrow 4$

$\Delta_{24} = CEF + BCEF$



path(35) = paths $\{1,2\} \leftrightarrow \{3,5\}$

$\Delta_{35} = F(BCDEF) + F(ABCDEF)$

Thm Have bijection $\phi : (\mathbb{R}^+)^{(k-1)(n-k-1)} \longrightarrow \underbrace{Gr_{k,n}(\mathbb{R}^+)/\text{torus}}_{\text{only parameterize interior region}}$

Def? $\text{Trop}^+ Gr_{k,n} = \text{Trop}^+ \widehat{Gr_{k,n}}$

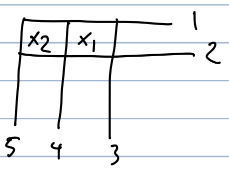
$= \bigcap_{f \in \text{plucker ideal}} \text{Trop}^+ V(f)$

Trop $\phi: \mathbb{R}^{(k-1)(n-k-1)} \longrightarrow \mathrm{Trop}^+ Gr$ / tropicalized torus action

$\mathrm{Sum}(S) = \sum_{\substack{\text{paths} \\ p \in S}} \sum_{\substack{\text{regions } r \\ \text{below } p}} x_r$

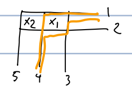
$\mathrm{Trop}\, \Delta_I = \min_{S \in \text{path } I} \mathrm{Sum}(S)$

<u>Ex</u>

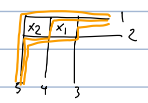


$\mathrm{Trop}\, \Delta_{23} = 0$

$\mathrm{Trop}\, \Delta_{24} = \min(0, x_1)$



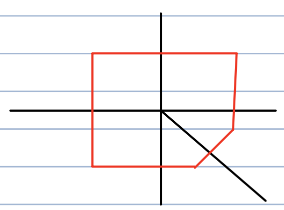
$\mathrm{Trop}\, \Delta_{25} = \min(0, x_1, x_1 + x_2)$



$\mathrm{Trop}\, \Delta_{34} = x_1$

$\mathrm{Trop}\, \Delta_{35} = \min(x_1, x_1 + x_2)$

$\mathrm{Trop}\, \Delta_{45} = x_1 + x_2$



<u>Observation</u>

$\mathrm{Trop}^+ Gr_{2,n}$ = normal fan of A_{n-3} associahedron