§ Valuated matroids

M a matroid on $[n]$
rank $M = d$

tropical numbers

$\mathcal{B} \subseteq \binom{[n]}{d}$ bases Think of $\mu : \binom{[n]}{d} \to \{0,\infty\} \subseteq \overline{\mathbb{T}} := \mathbb{R} \cup \{\infty\}$

$\mathcal{C} \subseteq 2^{[n]}$ circuits $C \in \{0,\infty\}^n$ $C(i) = \begin{cases} 0 & \text{if } i \in C \\ \infty & \text{if } i \notin C \end{cases}$

$V^* \subseteq 2^{[n]}$ covectors $V^* \in \{0,\infty\}^{[n]} \subseteq \overline{\mathbb{T}}^{[n]}$

orthogonal to every basis

Def

A <u>valuated matroid</u> is specified by the following

1) Basis valuation) A function
$\mu : \binom{[n]}{d} \to \overline{\mathbb{T}}$ that satisfies
any of the following (equivalent) axioms

a) (valuated symmetric basis exchange) $\forall\, B_1, B_2 \in \binom{[n]}{d}$
$\forall\, i \in B_1 \setminus B_2 \;\; \exists\, j \in B_2 \setminus B_1$
s.t. $\mu(B_1) + \mu(B_2) \geq \mu(B_1 - i + j) + \mu(B_2 - j + i)$

b) (Tropical Plücker relation)
$\forall\, I \in \binom{[n]}{d-1} \quad \forall\, J \in \binom{[n]}{d+1}$

$\min_{i \in J \setminus I} \{\mu(I+i) + \mu(J-i)\}$ is achieved at least twice

<u>Comment</u> One can think of μ as like tropical determinant (remembering $\mathrm{val}(0) = \infty$)

2) Valuated Circuits $\mathcal{C} \subseteq \overline{\mathbb{T}}^{[n]}$
<u>EX</u> $U_{3,4}$

$\mu(123) = 5$ $\quad$ 1234 is the only circuit in $U_{3,4}$
$\mu(124) = 1$
$\mu(134) = 1$ $\quad$ valuated circuit $\quad (0,1,1,5) + \mathbb{R}$ $\quad$ up to shifting
$\mu(234)\ 0$

$c_i := \mu(C - i)$

tropical hyperplane

$H_C :=$ bend locus $\quad$ $H_C :=$ bend locus of $\min\{x_1, x_2+1, x_3+1, x_4+5\}$

Def Let μ be a valuated matroid with valuated circuits $\mathcal{C}$ there is a tropical prevariety $\operatorname{Trop}\mu$ given by $\operatorname{Trop}\mu := \bigcap_{C \in \mathcal{C}} H_C$

Thm [MS 4.4.5] $\operatorname{Trop}\mu$ is a tropical variety of deg 1

(i.e. a tropical linear space)

Thm [Fink] All tropical linear spaces arise this way.
(i.e. If L is a trop variety of deg 1, then it is $\operatorname{Trop}\mu$ for some valuated matroid)

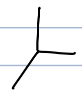
$$\left\{\begin{array}{l}\text{Tropical linear} \\ \text{spaces of dim } d-1 \text{ in } \mathbb{TP}^{n-1}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{valuated matroids} \\ \text{of rank } d \text{ on } [n]\end{array}\right\} \Big/ \begin{array}{l}\text{tropical} \\ \text{scaling}\end{array}$$

Example $U_{3,4}$

$H_C = \operatorname{trop} V \min\{x_1, x_2+1, x_3+1, x_4+5\}$

$\mu(123) = 5$
$\mu(124) = 1$
$\mu(134) = 1$
$\mu(234)\ 0$

EX $\min\{x_1, x_2, x_3\}$

 = bend locus of $V(\{x_1, x_2, 0\})$ $\Longrightarrow$ locus of non differentiability

$\uparrow$ tropical line

The set of all tropical linear spaces of dim $d-1$ in $\mathbb{T}\mathbb{P}^{n-1}$ is called the Dressian $Dr(d,n)$

(cut out by all tropical plucker relations)

is a prevariety in $\mathbb{T}\mathbb{P}^{\binom{n}{d}-1} \supseteq Dr(d,n)$

$$Gr(d,n) \xrightarrow{trop} Trop(Gr(d,n)) \subseteq Dr(d,n)$$

$$\cap \atop \mathbb{P}^{\binom{n}{d}-1}$$

(so we view $Gr(d,n)$ explicitly as its Plücker embedding)

Let V be a linear subspace given by its plücker vector $[V]$ $\xrightarrow{\text{coordinate wise valuation}}$ some valuated matroid

Rmk $Trop(Gr(2,n)) = Dr(2,n)$

$Trop(Gr(3,n)) \subsetneq Dr(3,n)$ when $n \geq 7$

$Trop(Gr^0(2,n)) =$ space of phylogenetic trees

Def A phylogenetic tree is a function on a trivalent tree (mod scaling)

$d: \binom{[n]}{2} \to \mathbb{R}$ $\quad d(i\,j) =$ distance between the leaf i and j

$n=4$