Structure theorem

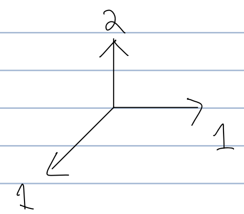
If $X \subseteq T^n$ is d-dim & irreducible

then $\mathrm{Trop}(X)$ is...

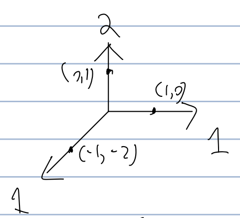
- a pure rational polyhedral complex
- weighted & balanced new terms today
- connected through codim 1

Def A weighted fan is a fan Σ which is pure of dim d & an assignment of weights $m(\sigma) \in \mathbb{N}$ for $\sigma \in \Sigma$ of dim d

EX



Can add vectors to each σ



$$\sum_\sigma m(\sigma) v_\sigma = 0$$

called the zero-tension condition

Def Let Σ be a weighted rational d-dim fan

for a cone $\tau \in \Sigma$ of dim $d-1$

minkowski difference

L = linear space parallel to τ $= \text{Span}(\tau - \tau)$

For $\sigma \in \Sigma$ s.t. $\sigma \supsetneq \tau$

then $(\sigma + L)/L$ is 1-dim in $\mathbb{R}^n/L$

Let v_σ be the fist lattice point on the ray

Then Σ is balanced at τ if $\sum_{\sigma \supsetneq \tau} m(\sigma) v_\sigma = 0$

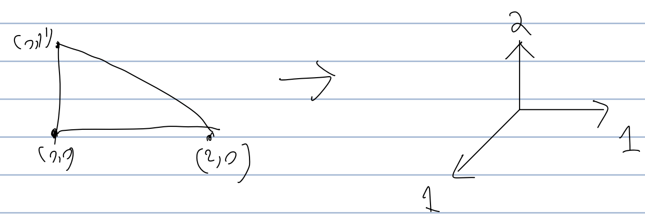
Σ is balanced if its balanced at all $\tau \in \Sigma$ of dim $d-1$.

A way to get balanced fans

- take normal fan of lattice polytope
- define lattice length(e) := (# lattice points in e -1)

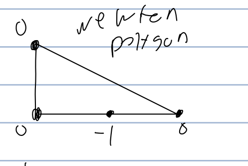
Ex

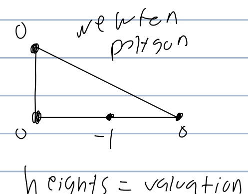
Newton polygon of
$x^2 + y + 1$



Process to go from puiseux series
to triangulated polygon

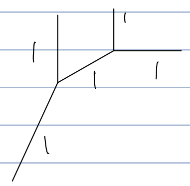
$x^2 + y + 1 + t^{-1}x \rightarrow$
$(2,0)$ $(1,0)$





take convex hull
& project down to
get triangulation

$x^2 + y + 1 + t^{-1}x \rightsquigarrow \text{Trop}$



take same lattice length for weights

Def A weighted polyhedral complex is pure of dim d & an assignment of weights $m(\sigma) \in \mathbb{N}$ for $\sigma \subset \Sigma$ of dim d

Def Σ = weighted polyhedral complex

Σ is balanced if $\text{star}_\Sigma(\tau)$ is balanced for all $\tau \in \Sigma$ with $\dim \tau = d-1$

$\text{Star}_\Sigma(\tau)$ = what's in Σ that touches τ

Let $F = \min_{\vec{u} \in P} (c_u + \vec{x} \cdot \vec{u})$ be a tropical poly

WLOG P is a lattice polytope

These $\{c_{\vec{u}}\}_{\vec{u} \in P}$ give a regular subdivision Δ of P

Tropical vanishing

Then $V(F)$ is the support of a polyhedral complex Σ_F

Each d-dim cone $\sigma \in \Sigma_F$ is dual to an edge $e(\sigma) \in \Delta$

Take $m(\sigma) :=$ lattice length of $e(\sigma)$

Def support of a polyhedral complex = union of cells of polyhedral complex

Prop 3.3.2 Σ_F is a balanced polyhedral complex

pf If $\dim(\tau) = d-1$

then τ is dual to a 2 dim face of Δ

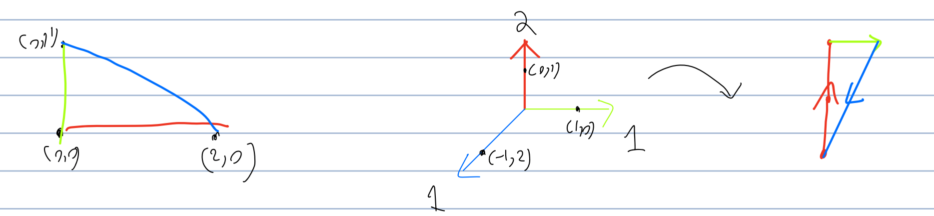
Let $L = \mathrm{span}(\tau - \tau)$

Then $\mathrm{star}(\tau)/L \underset{\text{canonical}}{\cong} \underset{\text{normal fan}}{N(Q)}$

Reduces to 2 dim case

Each $\sigma \supsetneq \tau$ is dual to an edge of Q.

and $m(\sigma)v_\sigma$ can be identified with $e(\sigma)$ rotated by $90°$



Sums to 0 b/c polygon is a cycle.

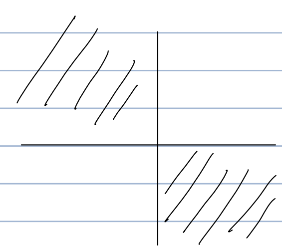
Def Σ = pure d-dim poly complex

A facet-ridge path is a sequence $P_1, \dots, P_s$ of d-dim cells

s.t. $P_i \cap P_{i+1}$ has $\dim = d-1$.

Σ is connected through codim 1 if $\forall\, P, P'$ there is a facet-ridge path from P to P'.

Non-Ex:



What if $X \subseteq T^n$, d-dim but not $d = n-1$?

<u>Def</u> If I is an ideal, then

$Ass(I) :=$ associated primes

$= \{ \sqrt{Q_i} \mid Q_i \in$ primary decomp of $I \}$

(Primary decomp of $I := \bigcap_i Q_i$ where $\sqrt{Q_i}$ is prime & irreducible & $\sqrt{Q_i}$ are distinct.

$Ass^{min}(I) =$ min elts in $Ass(I)$ (by inclusion)

$mult(P_i, I) := \ell\left((S/Q_i)_{P_i} \right)$ $\left(S = K[x_1^{\pm}, \ldots, x_n^{\pm}] \right)$

$\sigma =$ top-dim cell in $trop(V(I))$

$w \in relint(\sigma)$

$$mult(\sigma) = \sum_{P \in Ass^{min}(in_{\vec{w}}(I))} mult(P, in_{\vec{w}}(I))$$