the setting

Let K be an algebraically closed field with nontrivial valuation

$$\mathrm{val}: K \to \mathbb{R} \cup \{\infty\}$$

$$\mathrm{val}(K) =: \Gamma_{\mathrm{val}} \leq \mathbb{R}$$

"value group"

$\bar{a} :=$ image of $a \in R_K$ in $R_K/m_K = \Bbbk$

$\phi : \Gamma_{\mathrm{val}} \to K^*$ is the splitting $\phi(w) = t^w$, $\mathrm{val}(\phi(w)) = w$.

Ex. $K = \mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n}))$ field of Puiseux series.

The ring $R_K = \{c \in K : \mathrm{val}(c) \geq 0\}$ has a unique maximal ideal $m_K = \{c \in K : \mathrm{val}(c) > 0\}$, denote

$$\Bbbk = R_K / m_K$$ "residue field"

the goal

Reinforce that tropical geometry is a marriage between algebraic and polyhedral geometry.

↳ the tropical variety carries a polyhedral complex structure!

CONTENTS:

(i) Fix $f = \sum_i c_u x^u \in K[x^{\pm}] = K[x_1^{\pm}, \dots, x_n^{\pm}]$

$\hookrightarrow u := (u_1, \dots, u_n),\ u_i \in \mathbb{Z}.$

$x^u := x_1^{u_1} \cdots x_n^{u_n}$

def The tropicalization of f is given by

$$\mathrm{trop}(f)(w) = \min(\mathrm{val}(c_u) + w \cdot u) : \mathbb{R}^n \to \mathbb{R}$$

$\hookrightarrow u \cdot w = u_1 w_1 + \dots + u_n w_n$

perform additions/multiplications in tropical semiring

Ex. $K = \mathbb{C}\{\{t\}\}$, $f = x + y + 1.$

$$\mathrm{trop}(f)(w) = \min(w_1, w_2, 0)$$

Remark: The tropical polynomial is a piecewise linear function $\mathbb{R}^n \to \mathbb{R}$

Recall the classical variety of $f \in K[x^{\pm}]$ is the hypersurface in the algebraic torus $\mathbb{T}^n = (K^*)^n$ over K:

$$V(f) = \{v \in \mathbb{T}^n : f(v) = 0\}.$$

We now tropicalize this notion:

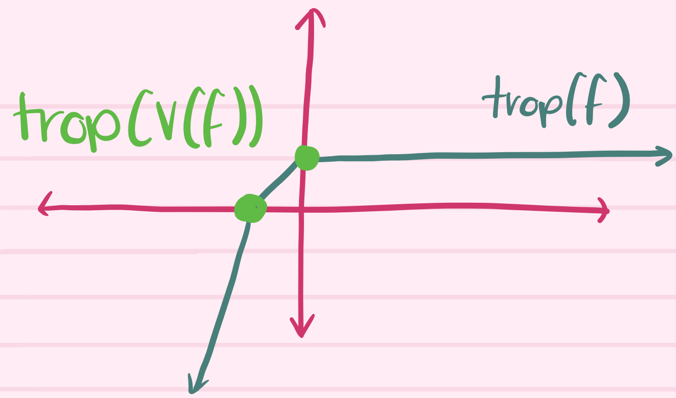
def The tropical hypersurface $\mathrm{trop}(V(f))$ is given by:

$$\mathrm{trop}(V(f)) = \{w \in \mathbb{R}^n : \text{minimum of } \mathrm{trop}(f) \text{ is achieved at least twice}\}$$

Let's interpret "achieved at least twice":

$K = \mathbb{Q}$, 2-adic valuation

$$f = 8x^3 + 2x + 2 \longrightarrow \mathrm{trop}(f) = \min\{3 + 3x, 1 + x, 1\}$$



Q: Where is $\min\{3+3x, 1+x, 1\}$ achieved twice?

A: $x = -1, 0$

In general, trop(V(f)) is the locus of points where the piecewise function trop(f) fails to be linear.

Recall also the following gadget:

$$\mathrm{in}_w(f) = \sum_{\substack{u :\, \mathrm{val}(c_u) + w\cdot u \\ = \mathrm{trop}(f)(w)}} \overline{t^{-\mathrm{val}(c_u)} c_u}\, x^u$$

<u>Remark</u>: The function of the normalizing factor $t^{-\mathrm{val}(c_u)}$ is to preserve all terms $c_u x^u$ for which $\mathrm{val}(c_u) + w\cdot u = \mathrm{trop}(f)(w)$ under $R_K \to R_K/m_K$

Such terms are the tropical analogue of leading terms with respect to a monomial term order in the classical Gröbner theory sense.

↳ the weight w (together with the valuation) control which terms are "leading"

def For I an ideal in $K[x^{\pm}]$, $\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle \subseteq \mathbb{k}[x^{\pm}]$

def For p a tropical polynomial, we we denote
$$V(p) = \{ w \in \mathbb{R}^n : \text{the minimum in } p(w) \text{ is achieved at least twice.} \}$$

(ii) <u>Thm</u> (Kapranov): Fix a Laurent polynomial $f = \sum_{u \in \mathbb{Z}^n} c_u x^u$ in $K[x^{\pm}]$. The following three sets coincide:

1. $\mathrm{trop}(V(f)) \subseteq \mathbb{R}$
2. $V(\mathrm{trop}(f)) = \{ w \in (\Gamma_{val})^n : in_w(f) \text{ is not a monomial} \} \subseteq \mathbb{R}^n$
3. $\{ (val(v_1), \ldots, val(v_n)) : v \in V(f) \} \subseteq \mathbb{R}^n$

Ex. Let $K = \mathbb{C}\{\{t\}\}$ and $f = x + y + 1 \in K[x^{\pm}, y^{\pm}]$

Then $\mathrm{trop}(f) = \min(x, y, 0)$, so

(1.) $= \{(a,0) : a \geq 0\} \cup \{(0,a) : a \geq 0\} \cup \{(-a,-a) : a \geq 0\}$ (*)

Let's compute (2.) and (3.).

(2.) Fix $a \in \mathbb{R}_{>0}$.

$w = (a, 0)$: $\mathrm{trop}(f)(w) = 0$ NOT A MONOMIAL!
$in_{(a,0)}(f) = y + 1$

$w = (0, a)$: $\mathrm{trop}(f)(w) = 0$ NOT A MONOMIAL!
$in_{(0,a)}(f) = x + 1$

$w = (-a, -a)$: $\mathrm{trop}(f)(w) = -a$ NOT A MONOMIAL!
$in_{(-a,-a)}(f) = x + y$

$w = (0,0)$: $\mathrm{trop}(f)(w) = 0$ NOT A MONOMIAL!
$in_{(0,0)}(f) = x + y + 1$

$(*) \subseteq (2.)$. Exercise: check $(2.) \subseteq (*)$

(3.) We have $V(f) = \{(x,y) \in (K^*)^2 : x+y+1=0\}$

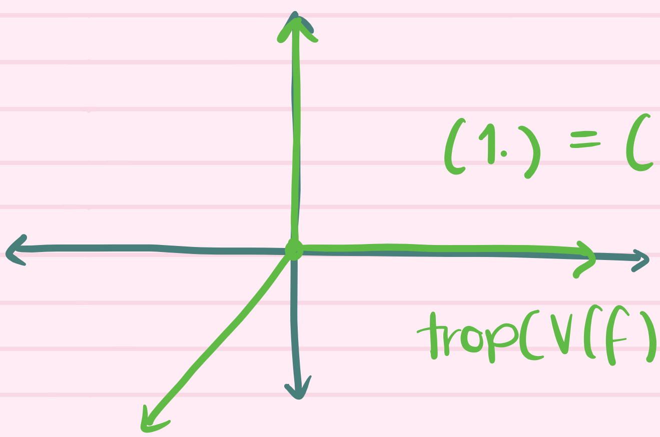
$$= \{(x,-1-x) : x \in K \setminus \{0,-1\}\}.$$

Then:

$$(\mathrm{val}(x), \mathrm{val}(-1-x)) = \begin{cases} (\mathrm{val}(x), 0) & \text{if } \mathrm{val}(x) > 0 \\ (\mathrm{val}(x), \mathrm{val}(x)) & \text{if } \mathrm{val}(x) < 0 \\ (0, a) & \text{if } x = -1 + \alpha t^a + \text{H.O.T.} \\ (0,0) & \text{else.} \end{cases}$$

Ranging x over $K \setminus \{0,-1\}$ and taking closure, we get

$$(*) = (3.)$$



$(1.) = (2.) = (3.)$

$\mathrm{trop}(V(f))$

<u>Pf (of thm)</u>:

$(1.) = (2.)$ $w \in \mathrm{trop}(V(f)) \Longleftrightarrow \underset{\mathrm{trop}(f)(w)}{\overset{\|}{\min\{\mathrm{val}(c_u) + u \cdot w\}}}$ achieved at least twice

$\Longleftrightarrow \mathrm{in}_w(f) = \sum_{\substack{u : \mathrm{val}(c_u) + u \cdot w \\ = \mathrm{trop}(f)(w)}} \cdots$ is not a monomial.

$(3.) \subseteq (1.)$ Note $\mathrm{trop}(V(f))$ is closed (we will see that it is the support of a polyhedral complex of dimension $(n-1)$).

So fix $(\mathrm{val}(v_1), \ldots, \mathrm{val}(v_n))$, $f(v) = 0$.

Recall:
Lemma 2.1.1 $\mathrm{val}(a) \neq \mathrm{val}(b) \Longrightarrow \mathrm{val}(a+b) = \min(\mathrm{val}(a), \mathrm{val}(b))$

Then $\mathrm{val}(\sum_i c_u v^u) = \mathrm{val}(f(v)) = \infty > \mathrm{val}(c_u v^u)$ $\forall u : c_u \neq 0$

$\Longrightarrow \min\{\mathrm{val}(c_u v^u)\} = \min\{\mathrm{val}(c_u) + u \cdot v\}$ achieved twice, else you could iteratively add minimum to every other term and retain the minimum value for $\mathrm{val}(f(v))$.

$(1.) \subseteq (3.)$ Omitted, see Proposition 3.1.5

Slogan: "zeros of initial forms lift to zeros of f"

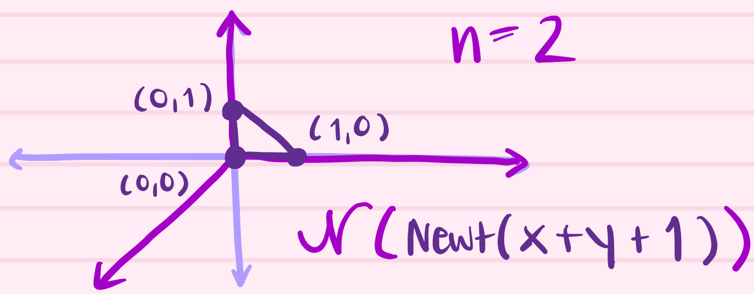
$\square$

Two final polyhedral geometric thoughts:

Proposition: For $f \in K[x^{\pm}]$ a Laurent polynomial, $\mathrm{trop}(V(f))$ is the support of a pure Γ_{val}-rational polyhedral complex of dimension $n-1$. It is the $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope of f given by weights $\mathrm{val}(c_u)$ on the lattice points in $\mathrm{Newt}(f)$.

a special case

Proposition: If $val(c_u) = 0$ for $\forall u$, then $trop(V(f))$ is the support of an $(n-1)$-dimensional polyhedral fan, which is the $(n-1)$-skeleton of the normal fan to the Newton polytope of f.



The vertices of the Newton polytope of are the exponent vectors.

Elements of the normal fan are those weight vectors (linear functionals) whose minimizers (leading terms in the initial form) lie along a face of $Newt(f)$, that is, no single term (which we identify by its exponent vector u) uniquely minimizes $u \cdot w$.

(iii) The Fundamental Theorem of tropical algebraic geometry

def Let I be an ideal in $K[x^{\pm}]$ and $X = V(I)$ be its variety in $\mathbb{T}^n$. The <u>tropicalization</u> $trop(X)$ is the intersection of all tropical hypersurfaces defined by $f \in I$:

$$trop(X) = \bigcap_{f \in I} trop(V(f)) \subseteq \mathbb{R}^n.$$

Remark: $V(I) = V(\sqrt{I})$, so $\mathrm{trop}(X)$ depends only on the radical ideal $\sqrt{I}$.

We call a "tropical variety" in $\mathbb{R}^n$ any subset of the form $\mathrm{trop}(X)$, where X is a subvariety of $\mathbb{T}^n$ for K with valuation.

Q: Can we realize $\mathrm{trop}(X)$ with finitely many intersections?

A: Yes!

def Let I be an ideal in $K[x^{\pm}]$. A finite generating set $\mathcal{T}$ of I is a <u>tropical basis</u> if $\forall w \in \mathbb{R}^n$, $\exists f \in I$ for which the minimum in $\mathrm{trop}(f)(w)$ is achieved only once iff $\exists g \in \mathcal{T}$ for which the minimum in $\mathrm{trop}(g)(w)$ is achieved only once.

↳ A finite collection which still captures an accurate tropical portrait.

↳ Tropical geometry concerned with weights $w \in \mathbb{R}^n$ for which $\mathrm{in}_w(I)$ is a proper ideal in $\mathbb{K}[x^{\pm}]$; a tropical basis captures this information

↳ Tropical bases are analogues of Gröbner bases in that both furnish finite sets of data encoding all ideal degenerations under "leading term" operations

Thm Every $I \subseteq K[x^{\pm}]$ admits a finite tropical basis.

Corollary: $\mathrm{trop}(X) = \bigcap_{f \in \mathcal{T}} \mathrm{trop}(V(f))$.

Ex. $K = \mathbb{C}\{\{t\}\}$, $I = \langle x+y+z, x+2y \rangle$. Then

$\mathrm{trop}(V(x+y+z)) = \{(x,y,z) \in \mathbb{R}^3 : x=y\leq z \text{ or } y=z\leq x \text{ or } x=z\leq y\}$

$\mathrm{trop}(V(x+2y)) = \{(x,y,z) \in \mathbb{R}^3 : x=y\}$

$$\Longrightarrow \operatorname{trop}(V(x+y+z)) \cap \operatorname{trop}(V(x+2y))$$
$$= \{(x,y,z) \in \mathbb{R}^3 : x = y \leq z\}$$

However, $(x+y+z)-(x+2y) = z-y \in I$, and

$$\operatorname{trop}(V(z-y)) = \{(x,y,z) \in \mathbb{R}^3 : y = z\}$$

but $(1,1,2) \notin \operatorname{trop}(V(z-y))$, hence we often need to consider intersections over an enlarged basis of I.

(intersect more than the given hypersurfaces)

So $\{x+y+z, x+2y\}$ is <u>NOT</u> a tropical basis:

$\operatorname{in}_{(1,1,2)}(\{x+y+z, x+2y\}) = \{x+y\}$ contains no monomials, while $\operatorname{in}_{(1,1,2)}(I) \ni y$.

<u>Thm</u> (Fundamental Theorem of Tropical Algebraic Geometry)

Let I be an ideal in $K[x^{\pm}]$ and $X = V(I)$ its variety in the algebraic torus $\mathbb{T}^n \cong (K^*)^n$. The following three subsets coincide:

1. $\operatorname{trop}(X)$
2. $\{w \in (\Gamma_{val})^n : \operatorname{in}_w(I) \neq \langle 1 \rangle\} \subseteq \mathbb{R}^n$
3. $\overline{\operatorname{val}(X)} = \overline{\{(\operatorname{val}(u_1), \dots, \operatorname{val}(u_n)) : (u_1, \dots, u_n) \in X\}} \subseteq \mathbb{R}^n$

Note $\operatorname{in}_w(I) = \langle 1 \rangle$ iff $\exists f \in I$, $\operatorname{in}_w(f)$ is a unit, i.e. a monomial, i.e. minimum of $\operatorname{trop}(f)(w)$ achieved at least twice.

thank you