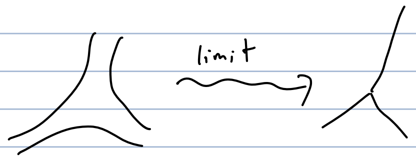
Gröbner Bases & Gröber Complexes

0) Motivation

$V(I) \mapsto$ Amoeba

$\log|V(I)|$

"Tropical variety"



Last time: Tropical variety $= \mathrm{Val}(V(I))$

Today

1) Background

Def. <u>Valuation</u> | k a field

$\mathrm{val}: k \to \mathbb{R} \cup \{\infty\}$ s.t.

(1) $\mathrm{val}(a) = \infty \iff a = 0$

(2) $\mathrm{val}(a \cdot b) = \mathrm{val}(a) + \mathrm{val}(b)$

(3) $\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$

<u>ex</u> p-adic val

$K = \mathbb{Q}$ $\quad p$ prime

$\mathrm{val}_p(r) = k$ if $r = p^k \frac{a}{b}$

with $p \nmid a$ & $p \nmid b$ $\quad \mathrm{val}_2\left(\frac{137}{8}\right) = -3$

$\mathrm{val}_2\left(\frac{4}{7}\right) = 2$

Def $R_{val} = \{ a \in K \mid val(a) \geq 0 \} \to$ local ring

$m_{val} = \{ a \in K \mid val(a) > 0 \} \to$ unique max ideal

residue field $:= R_{val} / m_{val} =: \mathbb{K}$

In ex) $\mathbb{R}_{val}$ = localization of $\mathbb{Z}$ on $\langle p \rangle$

$m_{val} = \{ \frac{a}{b} \mid p \mid a \ \& \ p \nmid b \}$

$R_{val} / m_{val} = \mathbb{Z}/p\mathbb{Z}$

Ex: Puiseux series

$\mathbb{C}\{\{t\}\} = \{$ formal power series

$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$ $\{a_i\}$ have bounded denominators

$a_1 < a_2 < \cdots \in \mathbb{Q}$

$val(c(t)) = a_1$

① Ex. $c(t) = \frac{1}{1-t} = 1 + t + t^2 + \cdots$ $val = 0$

$c(t) = \frac{1}{t^2 - t^3} = \frac{1}{t^2} \cdot \frac{1}{1-t}$ $val = -2$

$X_1(t) = \frac{1 + \sqrt{1-4t}}{2}$ $X_2 = \frac{1 - \sqrt{1-4t}}{2}$

$X_1(t) = 1 - \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{k+1} t^k$ $X_2 = \sum_{k=1}^{\infty} \frac{\binom{2k}{k}}{k+1} t^k$

$val = 0$ $val = 1$

Fact $\mathbb{C}\{\{t\}\}$ is algebraically closed

Γ_{val} = image of $val \subseteq \mathbb{R} \cup \{\infty\}$ called "valuation group"

lem If K is closed, $\exists$ hom $\varphi : (\Gamma_{val}, +) \longrightarrow (K^*, \cdot)$

s.t. $val(\varphi(w)) = w$

Notation $\varphi(w) = t^w$

EX $\psi(w) = t^w$ in $\mathbb{C}\{\{t\}\}$

Polyhedral geometry

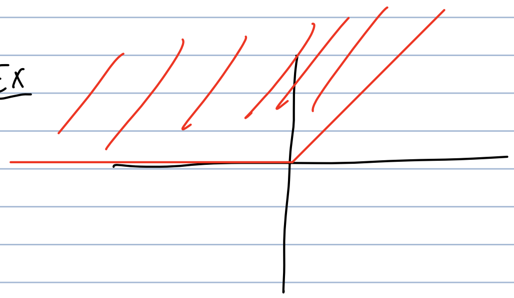
Def (polyhedron)

$\ell_i : \mathbb{R}^{n+1} \to \mathbb{R}$ linear

$\beta_i \in \mathbb{R}$ $\quad 1 \le i \le m$

$$P = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{c} \ell_i(x) \le \beta_i \\ 1 \le i \le m \end{array} \right\}$$

Ex



$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{array}{c} y \ge 0 \\ y \ge x \end{array} \right\}$$

Thm $A \subsetneq \mathbb{R}^{n+1}$ closed convex

$\exists\, \ell : \mathbb{R}^{n+1} \to \mathbb{R}$ & $\alpha \in \mathbb{R}$

$$A \subseteq \{ x \in \mathbb{R}^{n+1} \mid \ell(x) \le \alpha \}$$

Call $\{ x \in \mathbb{R}^{n+1} \mid \ell(x) = \alpha \}$ a "separating hyperplane"

Def Face of polyhedron

$F = P \cap H$

$\uparrow$ separating hyperplane

(2) Initial ideals

K: alg closed field with val

$f = \sum_{u \in \mathbb{N}^{n+1}} c_u \cdot x^u \in K[x_0, \dots, x_n]$

$x^u = x_0^{u_0} \cdots x_n^{u_n}$

$w \in \mathbb{R}^{n+1}$

Def Initial form

$$\mathrm{in}_w(f) \triangleq \sum c_u \cdot t^{-\mathrm{val}(c_u)} x^u$$

$$u \in \operatorname{argmin}_{c_u \neq 0} \{ \mathrm{val}(c_u) + w \cdot u \}$$

EX $f = t x_1^2 + 2x_1x_2 + 3t^2 x_2^2 + 4x_0x_1 + 5x_0x_2 + 6t x_0^2$

$w = (1, -1, 1)$

val(t)	1		2	0	
u	(0,2,0)	(0,1,1)	(0,0,2)	(1,1,0)	(1,0,1)
	1-2=-1	0+0=0	2+2=4	0+0	0+2

argmin = (0,2,0)

$\mathrm{in}_w(f) = (t \cdot t^{-1}) x_1^2 = x_1^2$

② $w = (2, 0, 1)$

$\mathrm{in}_w(f) = x_1^2 + 2x_1x_2$

Def Initial ideal

I homogeneous ideal

$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f), f \in I \rangle \subseteq K[x_0, \dots, x_n]$

$\triangle\ I = \langle f_1, f_2, \dots, f_s \rangle \not\Rightarrow \mathrm{in}_w(I) = \langle \mathrm{in}_w(f_1), \dots, \mathrm{in}_w(f_s) \rangle$

Def Grobner basis wrt w

$\{g_1, \dots, g_s\} \subseteq I$ is a grobner basis if

$\langle in_w(g_1), \dots, in_w(g_s) \rangle = in_w(I)$

If I is homogeneous then $\langle g_1, \dots, g_s \rangle = I$

Def $C_I[w] = \{ w \in \mathbb{R}^{n+1} \mid in_w(I) = in_w(I) \}$

$\overline{C_I[w]}$ = Euclidean closure of $C_I[w]$

Thm (1) $\overline{C_I[w]}$ is a polyhedron

(2) $\left| \{ \overline{C_I[w]} \mid w \in \mathbb{R}^{n+1} \} \right| < \infty$

a polyhedral complex

(3) $\overline{C_I[w]}$ is a max cell $\Longleftrightarrow$ $in_w(I)$ is a monomial

Ex $I = \langle f \rangle$ $w = (1,-1,1)$

$1 + 2w_i < 0 + w_1 + w_2$
$2 + w_2$

picture for example