K = alg. closed field w/ val: $K^* \to \mathbb{R}$

(e.g. $K = \overline{\mathbb{C}(t)}$ $k = \mathbb{C}$. $\mathrm{val}(f \in \mathbb{C}[t])$ is the order of vanishing at 0)

Fix polyn. ring $K[p]$ in $\binom{n}{d}$ variables

$$p = \{ p_I : I \in \binom{[n]}{d} \}$$ each variable indexed by d-subset of $[n]$

$I_{d,n}$ = Plücker ideal.

Def. The tropical Grassmannian $\mathcal{G}_{d,n}$:= the tropical variety $\mathcal{T}(I_{d,n})$

$$= \{ w \in \mathbb{R}^{\binom{n}{d}} : \mathrm{in}_w(I_{d,n}) \text{ contains no monomials} \}$$

Since $K[p]/I_{d,n}$ has krull. dim $(n-d)d+1$,

the tropical Grassmannian $\mathcal{G}_{d,n}$ is a polyhedral fan in $\mathbb{R}^{\binom{n}{d}}$

pure of dim $(n-d)d+1$. (structure thm. from Grant's talk)

Fix $i \in [n]$, denote $\mathbb{1}_i \in \mathbb{R}^{\binom{n}{d}}$ the vector

12 34 13 24 14 23

$$(\mathbb{1}_i)_I = \begin{cases} 1, & i \in I \\ 0, & \text{else} \end{cases}$$

Then $\forall\ w \in \mathbb{R}^{\binom{n}{d}}$, $w \in \mathcal{T}(I_{d,n}) \iff w + \mathbb{1}_i \in \mathcal{T}(I_{d,n})$

This is b/c $Gr(d,n)$ is torus-invariant

$\iff$ each i appears the same number of times in each monomial in any plücker relation

$\iff$ $\mathrm{in}_w(f) = \mathrm{in}_{w+\mathbb{1}_i}(f)$ $\forall\ f \in I_{d,n}$.

Thus. consider the linear map $\varphi : \mathbb{R}^n \hookrightarrow \mathbb{R}^{\binom{n}{d}}$ sending

$$e_i \mapsto \mathbb{1}_i$$

	1	2	3	4
12	1	1		
13	1		1	
14	1			1
23		1	1	
24		1		1
34			1	1

φ is obviously injective. ($\varphi(\sum a_i e_i) = 0$

$\Rightarrow \sum_{i \in I} a_i = 0$ $\forall\ I \in \binom{[n]}{d}$

$\Rightarrow a_i = a$ $\forall i$ $\Rightarrow a_i = 0$ $\forall i$.

Conclude that

$\mathrm{Im}\,\phi$ is an n-dim linear subspace of $\mathbb{R}^{\binom{n}{d}}$ contained in $\bigcap$ all cones in $\mathcal{G}_{d,n}$.

In particular, the all-1 vector $(1,1,\dots,1) \in \mathbb{R}^{\binom{n}{d}}$

$$= \frac{1}{d} \cdot \sum_{i\in[n]} \mathbb{1}_i \quad \in \mathrm{Im}\,\phi.$$

Def. • $\mathcal{G}'_{d,n} := \mathcal{G}_{d,n} / \mathrm{span}\{(1,1,\dots,1)\}$ is a polyhedral fan of dim $d(n-d)$ in $\mathbb{R}^{\binom{n}{d}} / \mathrm{span}\{(1\dots1)\}$

$d(n-d)+1-n$
$\quad\|$

• $\mathcal{G}''_{d,n} := \mathcal{G}_{d,n} / \mathrm{Im}\,\phi$ is a polyhedral fan of dim $(d-1)(n-d-1)$ in $\mathbb{R}^{\binom{n}{d}} / \mathrm{Im}\,\phi$.

• $\mathcal{G}'''_{d,n} = \mathcal{G}''_{d,n} \cap$ unit sphere is a polyhedral complex

Each maximal face is a polytope of dim $d(n-d)-n$ = this number -1.

Ex. $I_{2,4} = \langle p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \rangle$

$\mathcal{G}_{2,4} \subset \mathbb{R}^5$ consists of 3 cones $\mathbb{R}^4 \times \mathbb{R}_{\geq 0}$ glued along $\mathbb{R}^4 = \mathrm{im}\,\varphi$.

$C_1 = \{ w \in \mathbb{R}^6 : w_{12}+w_{34} = w_{13}+w_{24} \leq w_{14}+w_{23} \}$ is 5-dim

and contains $\mathrm{span}\left\{ \begin{matrix} e_{12}+e_{13}+e_{14}, & e_{12}+e_{23}+e_{24} \\ e_{13}+e_{23}+e_{34}, & e_{14}+e_{24}+e_{34} \end{matrix} \right\} = \mathrm{im}\,\varphi \cong \mathbb{R}^4$

after quotienting out $\mathbb{R}^4$-worth of $\mathrm{im}\,\varphi$,

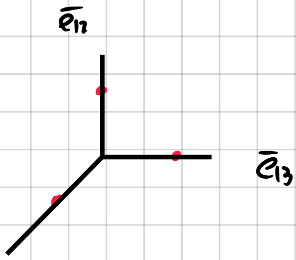
pick basis $\bar{e}_{12}, \bar{e}_{13}$.

$C_1 = \{ a\bar{e}_{12} + b\bar{e}_{13} : a = b \leq 0 \}$

$C_2 = \{ \qquad a = 0 \leq b \}$

$C_3 = \{ \qquad b = 0 \leq a \}$

$\rightarrow \mathcal{G}''_{2,4} =$



$\mathcal{G}'''_{2,4} = 3$ pts.

Thm [Speyer - Sturmfels]

The polyhedral complex $G''_{2,n}$ is the simplicial complex T_n of phylogenetic trees

$\#$ vertices $= 2^{n-1} - n - 1$

$\#$ facets $= 1 \cdot 3 \cdot 5 \cdots (2n-5)$

$G''_{2,n} \underset{\text{homotopy}}{\simeq} \bigvee (n-2)!$ spheres of dim $n-4$, minimal nonfaces are pairs of vertices. → flag complex

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pf. 1° Construct T_n as abstract simplicial complex,

2° realize it geometrically in $\mathbb{R}^{\binom{n}{2}} / \operatorname{im}\psi$.

3° show that it agrees w/ $G''_{2,n}$.

step 1°. • $\operatorname{Vert}(T_n)$ = set of unordered pairs $\{A, B\}$, where $A \sqcup B = [n]$, $|A|, |B| \geq 2$

$$|\operatorname{Vert}(T_n)| = 2^{n-1} - n - 1$$

• $E(T_n) = \{A, B\} - \{A', B'\}$ iff $A \subset A'$ or $A \subset B'$ or $B \subset A'$ or $B \subset B'$.

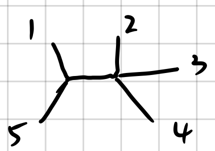
• T_n is a flag complex

i.e. $\sigma \subset \operatorname{Vert}(T_n)$ is a face iff

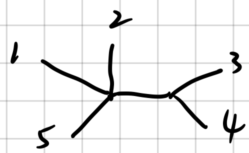
$\forall \{\{A,B\}, \{A',B'\}\} \subset \sigma$, $\{A,B\} - \{A',B'\} \in E(T_n)$.

• Each face σ is labeled by a semi-labeled tree on $[n]$ s.t. internal vertices have ≥ 3 edges.

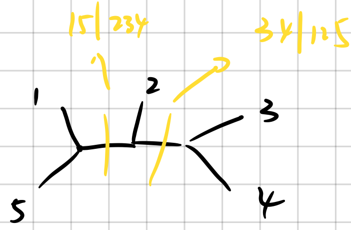
e.g.



labels the vertex $\{15, 234\}$



labels the vertex $\{125, 34\}$



labels the edge connecting them.

each internal edge gives a cut on $[n]$

facets $\overset{1:1}{\longleftrightarrow}$ semi-labeled trivalent trees on $[n]$, size $(2n-5)!!$

each facet has size $n-3$ = # internal edges of a trivalent tree on n ∂ vertices

$\Rightarrow \mathbb{T}_n$ is pure of dim $n-4$. (matching dim $G''_{2,n}$)

Step 2: We describe an embedding of $\mathbb{T}_n \hookrightarrow \mathbb{R}^{\binom{n}{2}} / \text{image } \phi$ $\longrightarrow$ $\frac{1}{2}n(n-3)$

by embedding maximal cones B_6, labeled by a trivalent tree 6

Def. A realization of the tree 6 is a CW-complex in $\mathbb{R}^?$ realizing the graph 6,

Given realization, define $d(i,j)$ = length of unique path from i to j.

index boundary vertices. pts $\hookleftarrow$ assigning length to edges

Set $B_6 = \{ (w_{ij}) \in \mathbb{R}^{\binom{n}{2}} : \exists$ realization of 6 s.t. $-w_{ij} = d(i,j)$ $\forall i,j \}$

$+ \operatorname{im} \varphi$ $\longrightarrow$ shouldn't worry about it too much

B_6 is a cone in $\mathbb{R}^{\binom{n}{2}}$

$C_6 = B_6 / \operatorname{im}\varphi$

adding $(\operatorname{im}\varphi)_{\geq 0}$ means adding length of "leaf edges"

2 assignments of edge lengths are identified if they only differ at "leaf edges"

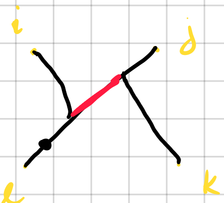
we add $\operatorname{im}\varphi$ so that we can take quotient

Thm. $\overline{C_6}$ is a simplicial cone of dim $|6|$ (# internal edges) w/ relative interior C_6.

$\{\overline{C_6}\}$, as 6 range through all trees, is a simplicial fan.

Claim: $\overline{B_6}$ is cut out by the 4-leaf condition:

$\forall i,j,k,l$. $\min\{ w_{ij} + w_{kl},\ w_{il} + w_{kj},\ w_{ik} + w_{jl} \}$ attains twice



① find unique path $l \to j$

② find unique path $l \to i$, $l \to k$, find the points where path divide

③ in this case, $d(i,k) + d(l,j) = d(i,j) + d(l,k) \geq d(i,l) + d(j,k)$

Thus, we showed that these conditions are necessary.

sufficiency is proved by explicitly constructing a tree from "Additive linkage algorithm".

$\Rightarrow \overline{B_6}, \overline{C_6}$ are simplicial cones.

Now, $\overline{B_6} = \mathrm{im}\,\varphi + \mathbb{R}_{\geq 0} \cdot \{ E_{A,B} \mid \{A,B\} \in 6 \}$,

where $E_{A,B} = \sum_{\substack{i \in A \\ j \in B}} e_{ij} \to$ basis vector in $\mathbb{R}^{\binom{n}{2}}$

i.e. assign positive length $E_{A,B}$ for each internal edge realizing the cut $A|B$.

b/c the edge $A|B$ appears in $d(i,j)$ iff

i,j is separated by this partition

Thus if $6, \tau$ are 2 trees

$6 \cap \tau$ is also a tree (b/c T_n is a simplicial complex)

from this, $\overline{B_6} \cap \overline{B_\tau} = \overline{B_{6\cap\tau}}$ $\boxtimes$

<u>Cor</u> We get a simplicial fan $\{\overline{C_6}\}$, pure of dim $n-3$.

Step 3: We need to show that this simplicial fan $= G_{2,n}''$.

We know that $I_{2,n}$ consists of quadrics $P_{ij}P_{kl} - P_{ik}P_{jl} + P_{il}P_{jk}$.

thus, vanishing of $V(\mathrm{trop}(\quad\checkmark\quad))$ is exactly the 4-leaf condition on i,j,k,l.

$\Rightarrow$ every relative open cone of $G_{2,n} \subseteq C_6$ for unique 6,

can be constructed using Additive Linkage algorithm.

To show that C_6's are actually a cone in $G_{2,n}$,

suffices to show it for 6 maximal faces (b/c $\overline{C_6} \cap \overline{C_\tau} = \overline{C_{6\cap\tau}}$)

By defn, this means that

fix any trivalent tree 6 and weight vector $w \in C_6$ (realized by assigning $\mathbb{R}_{>0}$ lengths to edges)

$\mathrm{in}_w(I_{2,n}) = J_6$

$\langle P_{ij}P_{kl} - P_{ik}P_{jl} \mid \{\{i,l\},\{j,k\}\}$ is a 4-leaf subtree of $6\}$

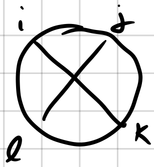
we know $\supseteq$

* We show that two ideals have a common initial monomial ideal w.r.t. some

($\{in f\}$ generate in I $\Rightarrow$ $\{f\}$ gens I)

monomial ordering.

Pick it to be $<_{circ}$

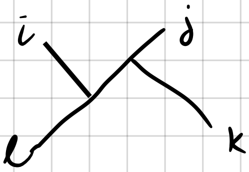


where cross terms dominate

turns out: $<_{circ}$ refines $<_w$

to see this, realize the trivalent tree G as a planar trivalent tree.

Then, for any 4-vertex subtree



$$[ij][kl] - [ik][jl] + [il][jk]$$

$\leq w$ $\underline{[ij][kl]}$, $\underline{[ik][jl]}$

$<_{circ}$ $\underline{[ik][jl]}$

so $in_{<_{circ}}(in_w(I_{2,n})) = in_{<_{circ}}(I_{2,n})$ $\underset{\text{classical result from invariant theory}}{=}$ $\langle [ik][jl] \rangle$

$\subseteq in_{<_{circ}}(J_n)$

$\Rightarrow in_w(I_{2,n}) = J_G$. $\square$

Note: generators in J_G has only ± 1 coefficients

$\Rightarrow G_{2,n}$ doesn't depend on char k.

map $w \in \mathcal{G}'_{d,n}$ to $L_w = \bigcap_{J \in \binom{[n]}{d+1}} V(\mathrm{Trop}(\sum_{j \in J} w_{J \setminus \{j\}} \cdot x_j))$

fan of dim $d(n-d)$ in $\mathbb{R}^{\binom{n}{d}} / \mathbb{R}(1,1,\dots,1)$

$\downarrow$

the tropical plane corresponding to a realizable valuated matroid

$d \, [\quad]$ matrix representing pt in $G(d,n)$

$x_1 \quad \overset{n}{-} \quad x_n$

condition: $(x_1 - x_n) \in$ row span.

Need to show: $w \mapsto L_w$ is injection.

To reconstruct $(w_{ij}) \in \mathbb{R}^{\binom{n}{d}} / \mathbb{R}(1,1 - 1)$,

suffices to reconstruct $w_{I \cup \{j\}} - w_{I \cup \{k\}}$ for $|I| = d-1$

from L_w.

plug in $x_j = t^M$ and obtained by solving linear system before tropicalization

Recipe: Fix large $M \in \mathbb{Q}$.

consider $\underbrace{L_w}_{\text{dim } d-1} \cap \underbrace{\{ x_i = M : i \in I \}}_{\text{dim } n-d+1}$ has a solution s.t. $x_j \leq M \quad \forall j \notin I$.

Then. $\sum_{\ell \in I \cup \{j,k\}} w_{I \cup \{j,k\} - \ell} \, x_\ell$ has leading term

$w_{I \cup j} \, x_k$ and $w_{I \cup k} \, x_j$

$\Rightarrow w_{I \cup j} - w_{I \cup k} = \underbrace{x_j - x_k}_{\text{coordinates of intersection}}$. $\square$