Chapter 1 of Sturmfels

Tropical geometry is the "combinatorial shadow" of AG

Def Tropical semiring

$(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

$x \oplus y = \min(x, y)$

$x \odot y = x + y$

Ex

$\forall x \quad x \oplus \infty = x$
$\forall x \quad x \odot 0 = x$

$x \odot \infty = \infty$

$x \oplus 0 = \begin{cases} 0 & x > 0 \\ x & x \leq 0 \end{cases}$

$3 \oplus 5 = 3$

$3 \odot 5 = 8$

Check that $x \odot (y + z) = x \odot y + x \odot z$

Fresh man's Dream

$(x \oplus y)^{\odot 3} = 3 \min(x \oplus y)$

$= \min(3x, 3y)$

$= x^{\odot 3} + y^{\odot 3}$

Also by distributivity

$(x \oplus y)^{\odot 3} = x^{\odot 3} \oplus x^{\odot 2} \odot y \oplus x \odot y^{\odot 2} \oplus y^{\odot 3}$

Def A tropical monomial is a product of variables

A tropical polymial is a tropical linear combination of trop monomials

Ex) $f(x) = x^{\odot 3} \oplus 1 \odot x^{\odot 2} \oplus 3 \odot x \oplus 6$

$f(x) = \min(3x, 2x+1, x+3, 6)$

Note $1 \odot x \neq x = 0 \odot x$

$$f(x) = \min(3x, 2x+1, x+3, 6) = \begin{cases} 3x & x \leq 1 \\ 2x+1 & 1 \leq x \leq 2 \\ x+3 & 2 \leq x \leq 3 \\ 6 & 3 \leq x \end{cases}$$

A tropical zero: A point at which the minimum is attained at least 2 times (by monomials in polynomial)

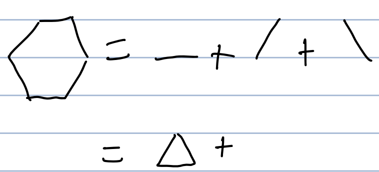
Tropical fundamental theorm of algebra: Every tropical polynomial function can be written uniquely as a product of linear terms

Ex $f(x) \stackrel{?}{=} (x \oplus 1) \odot (x \oplus 2) \odot (x \oplus 3)$

roots

Uniqueness won't hold in higher dimension

as



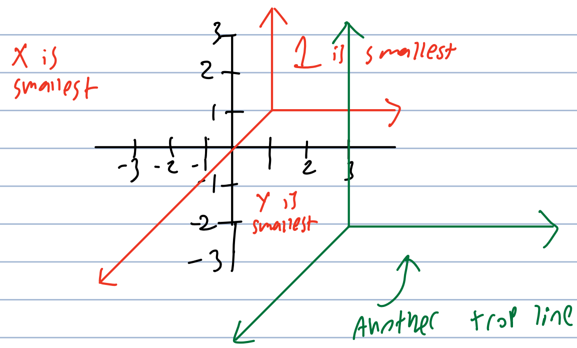
Plane curves Classically: $f(x,y)$ $V(f) = \{(x,y) : f(x,y) = 0\}$

Tropically: $p(x,y) = \bigoplus_{(i,j)} c_{ij} \odot x^{\odot i} \odot y^{\odot j}$

$V(p(x,y))$ = points in $\mathbb{R}^2$ s.t min is attained ≥ 2 times

$\boxed{\text{Ex}}$ Tropical line $\quad f(x,y) = x \oplus y \oplus 1$
$$= \min(x,y,1)$$

$$V(f) = \begin{cases} x = y \leq 1 \\ 1 = x < y \\ 1 = y < x \end{cases}$$



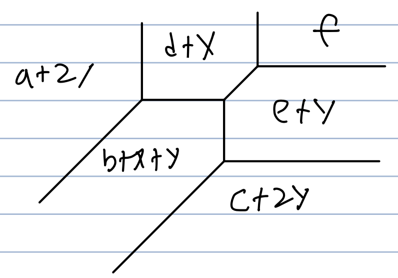
Thm 2 trop lines intersect in 1 point.

$\boxed{\text{Ex}}$ A tropical conic

$$a \odot x^2 \oplus b \odot x \odot y \oplus c \odot y^2 \oplus d \odot x \oplus e \odot y \oplus f$$

if $b+f < a+d \qquad d+f < b+e \qquad b+d < c+f$

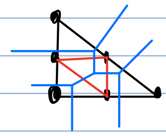
Then the curve looks like



Def Newton polygon: monomial $c_{ij} \odot x^i \odot y^j \longmapsto (i,j) \in \mathbb{R}^2$

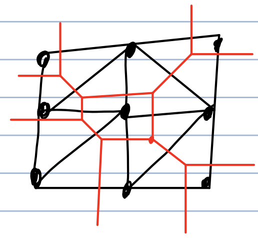
Newt(f) = convex hull of points coming from monomials of f.

Ex: $a \odot x^2 \oplus b \odot x \odot y \oplus c \odot y^2 \oplus d \odot x \oplus e \odot y \oplus f$



heights induce a subdivision

Triangulation is dual to $V(f)$



genus 1

Note: There is a tropical version of Bezout's theorem

The amoeba $\mathcal{A}(f)$ is the image of $V(f)$ under $(x,y) \mapsto (\log|x|, \log|y|)$

Ex: $f(x,y) = x+y+1$